PAPER *Special Section on Information Theory and Its Applications*

Miura's Generalization of One-Point AG Codes is Equivalent to Høholdt, van Lint and Pellikaan's Generalization

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SUMMARY Høholdt, van Lint and Pellikaan proposed a generalization of one-point AG codes, called the evaluation codes. We show that an evaluation code from a weight function can be constructed as Miura's generalization of one-point AG codes. Hence we can construct a one-point AG code as good as a given evaluation code from a weight function.

key words: *evaluation codes, one-point AG codes*

1. Introduction

Høholdt, van Lint and Pellikaan proposed a generalization of one-point AG codes called the evaluation codes, and enabled us to construct linear codes from arbitrary algebraic varieties[5], [6]. To construct an evaluation code, we find an appropriate K-algebra R with an order function w and an epimorphism $\varphi : R \to K^n$ of K-algebras, where K is a finite field and n is the code length. Then we define the evaluation code as a dual code of the image of some linear space of R under φ. When the order function w is a weight function, we have a lower bound for the minimum distance of an evaluation code that is a generalization of the Goppa designed minimum distance for algebraic geometry codes. Thus evaluation codes from a weight function are particularly useful, and it is interesting whether we can construct a good evaluation code from a weight function that is never obtained as an algebraic geometry code in terms of information rate and error-correcting capability.

In this paper we show that if a K-algebra R has a weight function then R is the affine coordinate ring of an affine algebraic curve with exactly one place at infinity, and the epimorphism $\varphi : R \to K^n$ is the evaluation of elements in R at K-rational points in the curve, hence the evaluation code from a weight function can be constructed as a linear code on an arbitrary algebraic curve that is proposed by Miura [9], [10]. In Miura's generalization of one-point AG codes, the author showed that we cannot construct a better linear code from a singular curve than the traditional one-point AG code on its normalization in terms of information rate and the Feng-Rao designed minimum distance [8], hence we can construct a one-point AG code as good as a given evaluation code from a weight function in terms of information rate and the Feng-Rao designed minimum distance.

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2. Order Functions and Evaluation Codes

In this section we review the evaluation codes. Let K be a finite field. A commutative ring R (and ψ_1) is said to be a *K-algebra* if there exists a ring monomorphism $\psi_1 : K \to R$. We regard the set K^n consisting of n-tuples of K as a commutative ring with the componentwise addition and multiplication. We define a ring monomorphism ψ_2:

$$K \longrightarrow K^n,$$
$$c \longmapsto (c, \ldots, c).$$

We regard K^n as a K-algebra with ψ_2. A ring homomorphism $\varphi : R \to K^n$ is said to be a *homomorphism of K-algebras* if $\psi_2 = \varphi \circ \psi_1$.

$\mathbf{N}_0$ denotes the set of nonnegative integers. A function $w : R \to \mathbf{N}_0 \cup \{-\infty\}$ is said to be an *order function* if it satisfies the conditions (O.0), (O.1), (O.2), (O.3) and (O.4) [6, Definition 3.4], and an order function is said to be a *weight function* if it satisfies (O.5) [6, Definition 3.5]:

(O.0) $w(f) = -\infty$ iff $f = 0$.
(O.1) $w(cf) = w(f)$ for all $c \in K \setminus \{0\}$.
(O.2) $w(f+g) \leq \max\{w(f), w(g)\}$ and equality holds if $w(f) \neq w(g)$.
(O.3) If $w(f) < w(g)$ and $h \neq 0$, then $w(fh) < w(gh)$.
(O.4) If $w(f) = w(g) \neq -\infty$, then there exists $c \in K$ such that $w(f - cg) < w(g)$.
(O.5) $w(fg) = w(f) + w(g)$, where the sum of an integer and $-\infty$ is $-\infty$.

If R is a K-algebra with an order function w, then there exists a K-basis $\{f_1, f_2, \ldots\}$ such that $w(f_i) < w(f_{i+1})$ [6, Proposition 3.12]. Suppose that $\varphi : R \to K^n$ is an epimorphism of K-algebras, then the evaluation code C_l is the dual code of the code generated by $\varphi(f_1), \ldots, \varphi(f_l)$. Then the minimum distance of C_l is not less than $d_\varphi(l) = \min\{\#\{(f_i, f_j) \mid w(f_i f_j) = w(f_{m+1})\} \mid m \geq l, C_m \neq C_{m+1}\}$ [6, Theorem 4.13].

When the order function w is a weight function and the set $\mathbf{N}_0 \setminus w(R \setminus \{0\})$ is finite, we have an analogue of the Goppa bound for an algebraic geometry code. Let $g = \#\mathbf{N}_0 \setminus w(R \setminus \{0\})$, then $d_\varphi(l) \geq l + 1 - g$ [6, Theorem 5.24]. We can correct $\lfloor (d_\varphi(l) - 1)/2 \rfloor$ or less errors by the Feng-Rao algorithm [6, Theorem 6.12]. In Example 11 we will show that we can sometimes correct errors more than $\lfloor (d_\varphi(l) - 1)/2 \rfloor$.

3. An Algebra with a Weight Function is a Curve

Let $\bar{K}$ be the algebraic closure of K, χ an affine algebraic curve defined over K. Then there exist a prime ideal I of the polynomial ring $K[X_1, \ldots, X_t]$ such that the ideal of χ is $I\bar{K}[X_1, \ldots, X_t]$. In this paper by the affine coordinate ring of χ, we mean $K[X_1, \ldots, X_t]/I$. Let R be the affine coordinate ring of χ, and F the quotient field of R. F/K is an algebraic function field of one variable. Let Q be a place of F/K. We say that χ *has exactly one place* Q *at infinity* if every nonzero element $f \in R$ has no pole other than Q.

Theorem 1: Let R be a commutative algebra over a finite field K, and w a weight function of R. We assume that there exists $x \in R$ such that $w(x) > 0$.

Then the quotient field F of R is an algebraic function field of one variable over K, and there exists a unique discrete valuation v of F/K and a unique positive integer d such that $w(x) = -dv(x)$ for all $x \in R$. Let Q be the place corresponding to v. Then the degree of Q is one, and R is the affine coordinate ring of an affine algebraic curve defined over K with exactly one place Q at infinity.

Corollary 2: Let R be a K-algebra. If w_1, w_2 are the weight functions of R, then there exists a positive rational number e such that $ew_1 = w_2$.

We divide the proof of the theorem above into several lemmas.

Lemma 3: [9, Lemma 5.2] Let M be a nonzero submonoid of the monoid of nonnegative integers $\mathbf{N}_0$. Then there exists a finite subset $\{a_1, \ldots, a_t\}$ generating M, that is, for any $m \in M$, m can be written as

$$m = n_1 a_1 + \cdots + n_t a_t,\ n_i \in \mathbf{N}_0.$$

Proof: Choose $t \in M \setminus \{0\}$. For $i = 1, \ldots, t$ we define a_i to be the minimum elements in $M \cap \{i + tn \mid n \in \mathbf{N}_0\}$. If $M \cap \{i + tn \mid n \in \mathbf{N}_0\} = \emptyset$, we define a_i to be 0. Then $a_1, \ldots, a_t$ generates M. □

Lemma 4: Let R and w be as in Theorem 1. R is finitely generated as an algebra over K.

Proof: Let $\{a_1, \ldots, a_t\}$ generates the monoid $w(R \setminus \{0\})$. Then there exist elements $f_1, \ldots, f_t$ such that $w(f_i) = a_i$. We claim $R = K[f_1, \ldots, f_t]$.

Choose nonzero $x \in R$. Then there exists $f_1^{n_1} \cdots f_t^{n_t}$ having the same weight as x. By the definition of the weight function, there exists $c \in K$ such that

$$w(x - cf_1^{n_1} \cdots f_t^{n_t}) < w(x).$$

Repeating the same argument on $x - cf_1^{n_1} \cdots f_t^{n_t}$, we get

$$w(x - \sum c_{n_1, \ldots, n_t} f_1^{n_1} \cdots f_t^{n_t}) < 0,$$

which implies

$$x = \sum c_{n_1, \ldots, n_t} f_1^{n_1} \cdots f_t^{n_t}.$$

□

Example 5: [4] In contrast to the weight function, we can construct a K-algebra with an order function that is not finitely generated over K. Let R be the K-linear subspace of $K[X, Y]$ generated by $\{X^i Y^j \mid j = 0 \text{ or } (j > 0 \text{ and } i > 0)\}$. Then R is a subring of $K[X, Y]$ and not finitely generated over K. Since $K[X, Y]$ has an order function, R has an order function.

Lemma 6: [6, Lemma 5.6] Let R, w be as in Theorem 1, $f \in R \setminus K$, and (f) be the ideal of R generated by f. Then the residue class ring $R/(f)$ is a nonzero finite-dimensional K-linear space.

Lemma 7: Let R, w be as in Theorem 1. R is an integral domain [6, Proposition 3.10]. Let F be the quotient field of R. F is an algebraic function field of one variable over K.

Proof: Since R is finitely generated over K, it is enough to show that the transcendence degree of F over K is one, which equals to the Krull dimension of the ring R [2, Theorem A, Chapter 13].

For $f \in R \setminus K$, $R/(f)$ is of Krull dimension 0 by the previous lemma, thus R is of Krull dimension 1 [2, Corollary 13.11]. □

Lemma 8: Let I be a prime ideal of $K[X_1, \ldots, X_t]$ and suppose that the Krull dimension of $R := K[X_1, \ldots, X_t]/I$ is 1. Let $\bar{R}$ is the integral closure of R in its quotient field. Then $\bar{R}$ modulo R is a finite-dimensional K-linear space.

Proof: $\bar{R}$ is finitely generated as an R-module [2, Corollary 13.13]. Thus there exists a prime ideal $\bar{I}$ of $K[X_1, \ldots, X_u]$ such that $\bar{R} = K[X_1, \ldots, X_u]/\bar{I}$. Let A be the annihilator of the R-module $\bar{R}$ modulo R, that is, $\{x \in R \mid x\bar{R} \subseteq R\}$. Then A is a nonzero ideal of $\bar{R}$ [7, Chapter X]. Let J be the ideal of $K[X_1, \ldots, X_u]$ containing $\bar{I}$ such that $J \bmod \bar{I} = A$. Then the number of zeros of J in the algebraic closure of K is finite, so the vector space dimension of $K[X_1, \ldots, X_u]/J = \bar{R}/A$ is finite [1, Theorem 2.2.7]. Hence $\bar{R}$ modulo R is finite-dimensional. □

Proof of Theorem 1: Let d be the greatest common divisor of $w(R) \setminus \{0, -\infty\}$. We define a function

v from F such that $v(f/g) = \frac{w(g)-w(f)}{d}$ for nonzero $f, g \in R$ and $v(0) = \infty$. Then v is a discrete valuation of the algebraic function field F/K in the sense of Stichtenoth [12, Definition I.1.9]. Let O_Q be the valuation ring corresponding to v, and Q the place of O_Q. By the definition (O.4) of the weight function, the residue field O_Q/Q is isomorphic to K, so the degree of Q is one.

Let $\mathbf{P}_F$ be the set of places in F/K, $\bar{R}$ the integral closure of R in F, and $S(R) := \{P \in \mathbf{P}_F \mid O_P \supset R\}$. Then $\bar{R} = \bigcap_{P \in S(R)} O_P$ [12, Theorem III.2.6]. We will show that $S(R) = \mathbf{P}_F \setminus \{Q\}$. Let $W := \{x \in \bar{R} \mid v_Q(x) > 0\}$. Then W is a K-linear space. Since the vector space dimension of $\bar{R}$ modulo R is finite and $R \cap W = \{0\}$, $\dim W < \infty$. Suppose that there exists $T \in \mathbf{P}_F \setminus (S(R) \cup \{Q\})$. By the strong approximation theorem of the discrete valuations [12, Theorem I.6.4], we can find $x_i \in F$ for $i = 1, 2, \ldots$ such that $v_Q(x_i) = i$ and $v_P(x_i) \geq 0$ for all $P \in \mathbf{P}_F \setminus \{T\}$. Then $x_i \in \bigcap_{P \in S(R)} O_P = \bar{R}$ and $x_i \in W$. $x_1, x_2, \ldots$ are linearly independent over K, which contradicts $\dim W < \infty$.

Therefore we conclude $S(R) = \mathbf{P}_F \setminus \{Q\}$, which implies that R is the affine coordinate ring of an affine algebraic curve with exactly one place Q of degree one at infinity.

Recall that $w = -dv$ as a function from R. We will show the uniqueness of d and v $(= v_Q)$. Suppose that there exist a positive integer d_1 and a discrete valuation v_1 of F/K such that $w(x) = -d_1 v_1(x)$ for all $x \in R$. Then by the definition of the discrete valuation [12, Definition I.1.9] there exist $f, g \in R \setminus \{0\}$ such that $1 = v_1(f/g) = v_1(f) - v_1(g) = (w(f) - w(g))/d_1$. So d_1 must be the greatest common divisor of $w(R) \setminus \{0, -\infty\}$ and $d_1 = d$. Let Q_1 be the place of v_1. Then $Q_1 \in \mathbf{P}_F \setminus S(R)$ and $Q_1 = Q$. □

4. Evaluation Codes from a Weight Function can be Constructed as Miura's Generalization of One-Point AG Codes

Miura [9], [10] generalized the construction of the improved one-point AG codes [3] to an arbitrary singular curve with exactly one place of degree one at infinity. In this section we review Miura's generalization of one-point AG codes, then we show that evaluation codes from a weight function can be constructed as Miura's generalization.

Let R be the affine coordinate ring of an affine algebraic curve defined over K with exactly one place Q of degree one at infinity. Miura's generalization of one-point AG codes is as follows [9], [10]. v_Q denotes the discrete valuation at Q, and there exists a K-basis of R

$$\{f_1, f_2, \ldots\} \tag{1}$$

such that $v_Q(f_i) > v_Q(f_{i+1})$. Let $M_1, \ldots, M_n$ be the pairwise distinct maximal ideals of R such that $R/M_i = K$ for $i = 1, \ldots, n$. We define the homomorphism of K-algebras τ as

$$\begin{array}{ccc} R & \longrightarrow & K^n, \\ f & \longmapsto & (f \bmod M_1, \ldots, f \bmod M_n). \end{array} \tag{2}$$

τ is an epimorphism of K-algebras, because by the Chinese Remainder Theorem the image of τ is isomorphic to $R/M_1 \times \cdots \times R/M_n$, which is isomorphic to K^n.

We define the subset $\{g_1, \ldots, g_n\}$ of the basis (1) inductively as follows. $g_1 = f_1$. g_i is defined as the element f_j having the smallest index such that $\tau(f_j)$ is linearly independent of $\tau(g_1), \ldots, \tau(g_{i-1})$. Choose a nonempty proper subset $B \subset \{g_1, \ldots, g_n\}$. The linear code $C(B)$ is defined as the dual code of the linear space generated by $\tau(B)$. It is a generalization of the improved one-point AG codes proposed by Feng and Rao [3], and if we define a weight function $w(f) = -v_Q(f)$ and take B as the first r elements of $\{g_1, g_2, \ldots, g_n\}$, then $C(B)$ is an evaluation code.

Let $B_0 = \{0\}$ and B_i the linear space generated by $\tau(g_1), \ldots, \tau(g_i)$. We define $M(s) := \{(g_i, g_j) \mid \tau(g_i g_j) \in B_s \setminus B_{s-1}$ and for $1 \leq u \leq i$, $1 \leq v \leq j$, $(u, v) \neq (i, j)$, $\tau(g_u g_v) \in B_{s-1}\}$ and

$$d_M(B) := \min\{\#M(s) \mid g_s \notin B\}.$$

Then the minimum distance of $C(B)$ is not less than $d_M(B)$ and we can correct $\lfloor (d_M(B) - 1)/2 \rfloor$ or less errors by the Feng-Rao algorithm. A fast error-and-erasure decoding algorithm is also available [11]. If we define $g = \#(\mathbf{N}_0 \setminus -v_Q(R \setminus \{0\}))$, then $\#M(s) \geq s - g$, which corresponds to the Goppa designed minimum distance of an algebraic geometry codes. If the affine curve corresponding to R is nonsingular, then g equals to the genus of the curve.

To show that an evaluation code can be obtained in this way, it is enough to show that if $\varphi : R \to K^n$ is an epimorphism of K-algebras, then φ is of the form (2).

Theorem 9: Let R be a K-algebra and $\varphi : R \to K^n$ is an epimorphism of K-algebras. Then there exists distinct maximal ideals $M_1, \ldots, M_n$ such that $R/M_i = K$ and φ equals to

$$\begin{array}{l} R \longrightarrow K^n, \\ x \longmapsto (x \bmod M_1, \ldots, x \bmod M_n). \end{array}$$

Proof: Let ρ_i be

$$\begin{array}{l} K^n \longrightarrow K, \\ (c_1, \ldots, c_n) \longmapsto c_i, \end{array}$$

for $i = 1, \ldots, n$. Let $M_i := \ker(\rho_i \circ \varphi)$. Then $R/M_i = K$ for $i = 1, \ldots, n$ and φ equals to

$$\begin{array}{l} R \longrightarrow K^n, \\ x \longmapsto (x \bmod M_1, \ldots, x \bmod M_n). \end{array}$$

$M_1, M_2, \ldots, M_n$ are pairwise distinct, otherwise φ is

not an epimorphism. □

We will clarify the relation between d_φ and d_M when $B = \{g_1, g_2, \ldots, g_r\}$.

Theorem 10: Let R be the affine coordinate ring of an affine algebraic curve with exactly one place Q of degree one at infinity, $w = -v_Q$ a weight function of R, $\{f_1, f_2, \ldots\}$ a K-basis of R such that $w(f_i) < w(f_{i+1})$, and $M_1, \ldots, M_n$ maximal ideals of R whose residue fields are K. We define φ as (2) and $\{g_1, \ldots, g_n\}$, d_φ, d_M as above. We choose B as the first r elements of $\{g_1, \ldots, g_n\}$ and suppose that $f_l = g_r$. Then $d_M(B) \geq d_\varphi(l)$.

Proof: We will show that if $w(f_i f_j) = w(f_{m+1})$ and $C_m \neq C_{m+1}$ then $f_i, f_j \in \{g_1, \ldots, g_n\}$. Suppose that $f_i \notin \{g_1, \ldots, g_n\}$. Then $\varphi(f_i)$ can be written as a linear combination of $\varphi(f_1), \ldots, \varphi(f_{i-1})$, which implies $\varphi(f_{m+1})$ can be written as a linear combination of $\varphi(f_1), \ldots, \varphi(f_m)$ that contradicts $C_m \neq C_{m+1}$.

Thus $\{(f_i, f_j) \mid w(f_i f_j) = w(f_{m+1})\} \subseteq \{(g_i, g_j) \mid \varphi(g_i g_j) \in B_s \setminus B_{s-1}$ and for $1 \leq u \leq i$, $1 \leq v \leq j$, $(u, v) \neq (i, j)$, $\varphi(g_u g_v) \in B_{s-1}\}$ with $f_{m+1} = g_s$. Therefore $d_M(B) \geq d_\varphi(l)$. □

The reader might think that $d_\varphi = d_M$. But we can construct an (abnormal) example such that $d_\varphi < d_M$.

Example 11: Let $\mathbf{F}_4$ be the finite field with 4 elements, and consider $R = \mathbf{F}_4[X, Y]/(Y^5 + Y + X^6)$. $x, y \in R$ denote the elements represented by X, Y. The affine algebraic set defined by $Y^5 + Y + X^6$ is an affine algebraic curve with exactly one place Q of degree one at infinity, and $v_Q(x) = -5, v_Q(y) = -6$ [9], [10]. We define the weight function $w = -v_Q$. Then $\{f_1, f_2, \ldots, f_{10}, \ldots\} = \{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\}$ is a $\mathbf{F}_4$-basis of R whose elements have pairwise distinct weights. The curve has 8 $\mathbf{F}_4$-rational points in the affine plane, and let $M_1, \ldots, M_8$ be the corresponding maximal ideals of R. We define φ as $R \to \mathbf{F}_4^8$, $x \mapsto (x \bmod M_1, \ldots, x \bmod M_8)$. Then $\{g_1, \ldots, g_8\} = \{1, x, y, x^2, xy, y^2, x^2y, y^3\}$. If we take B as the first r elements, then d_φ and d_M are as follows.

l	1	2	3	4	5	6	8
r	1	2	3	4	5	6	7
$d_\varphi(l)$	2	2	3	3	3	6	8
$d_M(B)$	2	2	3	4	5	6	8

5. Conclusion

Miura [9], [10] and Høholdt, van Lint and Pellikaan [5], [6] generalized one-point AG codes in two different ways, and the former seems to be a special case of the latter. In this paper we showed that they are same. Hence we can construct a one-point AG code as good as a given evaluation code from a weight function in terms of information rate and the Feng-Rao designed minimum distance.

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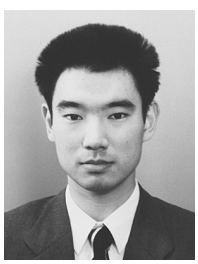
References

[1] W.W. Adams and P. Loustaunau, "An introduction to Gröbner bases," Graduate Studies in Mathematics, vol.3, American Mathematical Society, 1994.
[2] D. Eisenbud, "Commutative algebra with a view toward algebraic geometry," Graduate Texts in Mathematics, vol.150, Springer-Verlag, Berlin, 1995.
[3] G.L. Feng and T.R.N. Rao, "Improved geometric Goppa codes part I, basic theory," IEEE Trans. Inf. Theory, vol.41, no.6, pp.1678–1693, Nov. 1995.
[4] O. Geil and R. Pellikaan, "On the existence of order domains," preprint, 1999.
[5] T. Høholdt, J.H. van Lint, and R. Pellikaan, "Order functions and evaluation codes," Proc. AAECC-12, Lecture Notes in Computer Science, vol.1255, pp.138–150, Springer-Verlag, 1997.
[6] T. Høholdt, J.H. van Lint, and R. Pellikaan, "Algebraic geometry codes," in Handbook of Coding Theory, eds. V. Pless and W.C. Huffman, pp.871–961, Elsevier, 1998.
[7] S. Lang, Algebra, 3rd ed., Addison-Wesley, 1993.
[8] R. Matsumoto, "Linear codes on nonsingular curves are better than those on singular curves," IEICE Trans. Fundamentals, vol.E82-A, no.4, pp.665–670, April 1999.
[9] S. Miura, Ph.D. thesis, Univ. Tokyo, May 1997.
[10] S. Miura, "Linear codes on affine algebraic curves," IEICE Trans., vol.J81-A, no.10, pp.1398–1421, Oct. 1998.
[11] S. Sakata, D.A. Leonard, H.E. Jensen, and T. Høholdt, "Fast erasure-and-error decoding of algebraic geometry codes up to the Feng-Rao bound," IEEE Trans. Inf. Theory, vol.44, no.4, pp.1558–1564, July 1998.
[12] H. Stichtenoth, Algebraic function fields and codes, Springer-Verlag, Berlin, 1993.



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